

SEMIINFINITE COHOMOLOGY OF ASSOCIATIVE ALGEBRAS AND BAR DUALITY

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1. INTRODUCTION

Semiinfinite cohomology of graded associative algebras was first introduced in [Ar]. The setup for the definition includes a graded associative algebra A and two its graded subalgebras B and N such that N is positively graded, N is nonpositively graded and $A = B \otimes N$ as a graded vector space.

Basic examples of this situation are provided by universal enveloping algebras over graded Lie algebras. Let \mathfrak{g} be a graded Lie algebra, and $A := U(\mathfrak{g})$. Then set $\mathfrak{b} := \bigoplus_{n \leq 0} \mathfrak{g}_n$, $\mathfrak{n} := \bigoplus_{n > 0} \mathfrak{g}_n$, B and N are the universal enveloping algebras of the corresponding Lie algebras. The PBW Theorem provides the *triangular decomposition*: $U(\mathfrak{g}) \cong U(\mathfrak{b}) \otimes U(\mathfrak{n})$.

Lie algebra semiinfinite cohomology has a much longer history. It appeared first in mathematics over 10 years ago (see [F]). A proper homological construction for it belongs to Voronov (see [V]). Voronov treats Lie algebra semiinfinite cohomology as an exotic derived functor of a functor that is neither left nor right exact.

A similar approach for associative algebra semiinfinite cohomology was presented in [Ar]. Let k be a base field. Consider a “right semiregular A -module” $S_A := \text{Ind}_N^A \text{Coind}_k^N(k)$ and its endomorphism algebra $\text{End}_A(S_A) =: A^\sharp$. Suppose A^\sharp is augmented, i. e. there is a two-sided ideal $\overline{A}^\sharp \in A^\sharp$ such that the quotient algebra is k , let \underline{k} denote the trivial left A^\sharp -module. Then semiinfinite cohomology with coefficients in a graded A -module can be viewed as a “two sided derived functor” of the functor $\text{Hom}_{A^\sharp}(\underline{k}, S_A \otimes_A *)$ (see [V], [Ar]).

1.1. This paper is devoted to an alternative construction of associative algebra semiinfinite cohomology in terms of graded associative algebras’ bar duality. Recall the idea of the classical bar duality construction (see e. g. [BGS]). Let A be a nonnegatively graded associative algebra such that $\dim A_n < \infty$ and $A_0 = k$. Then one can define a canonical DG-algebra A^\vee and a functor $D : A\text{-mod} \longrightarrow A^\vee\text{-mod}$ defined on suitably chosen derived categories such that:

- D is an equivalence of triangulated categories;
- $H^\bullet(A^\vee) = \text{Ext}_A^\bullet(\underline{k}, \underline{k})$;
- $H^\bullet(A^{\vee\vee}) = A$;
- $H^\bullet(D(M)) = \text{Ext}_A^\bullet(\underline{k}, M)$ for a graded A -module M ;
- $H^\bullet(D^{-1}(N)) = \text{Tor}_\bullet^{A^\vee}(\underline{k}, N)$ for a A^\vee DG-module N satisfying certain grading conditions.

Recall also that $A^{\text{opp}\vee} \cong A^{\vee\text{opp}}$.

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1.2. The construction of associative algebra semiinfinite cohomology is a direct generalization of the described one.

Suppose we have a graded associative algebra A and its nonpositively graded subalgebra B satisfying certain finiteness conditions. Suppose also that there is a left B -augmentation of A , i. e. a left A -module \underline{B} such that when restricted to B the module \underline{B} is isomorphic to the left regular representation of B . Note that a triangular decomposition $A = B \otimes N$ provides a natural left B -augmentation of A : $\underline{B} := A \otimes_N \underline{k}$. In the second section we construct a canonical DG-algebra $D^\bullet(A, B)$ and functors $D_A^\downarrow : A\text{-mod} \longrightarrow D^\bullet(A, B)^{\text{opp}}\text{-mod}$, $D_A^\uparrow : A^{\text{opp}}\text{-mod} \longrightarrow D^\bullet(A, B)\text{-mod}$ defined on suitably chosen derived categories of modules such that

- D_A^\downarrow and D_A^\uparrow are equivalences of triangulated categories;
- $H^\bullet(D^\bullet(A, B)) = \text{Ext}_A^\bullet(\underline{B}, \underline{B})$;
- $H^\bullet(D_A^\uparrow(M)) = \text{Tor}_\bullet^A(\underline{B}, M)$ for a right A -module M .
- $H^\bullet(D_A^\downarrow(M)) = \text{Ext}_A^\bullet(\underline{B}, M)$ for a left A -module M .

However it is no longer true that taking the opposite algebra commutes with duality. The general case is as follows. There is another graded associative algebra A^\sharp containing B^{opp} as a subalgebra (and having a triangular decomposition $A^\sharp = B^{\text{opp}} \otimes N^{\text{opp}}$ as a graded vector space — if A has one) such that $D^\bullet(A, B)^{\text{opp}} \cong D^\bullet(A^\sharp, B^{\text{opp}})$.

In the third section we define semiinfinite Ext functor as follows. For a graded A -module M and a graded A^\sharp -module L satisfying certain grading conditions

$$\text{Ext}_A^{\frac{\infty}{2}+\bullet}(L, M) := \text{Ext}_{D^\bullet(A, B)}^\bullet(D_A^\uparrow(L), D_A^\downarrow(M)).$$

We prove also that the algebra A^\sharp up to a completion coincides with the algebra $\text{End}_A(S_A)$ where $S_A := \text{Hom}_B(A, B)$. Note that in the case of triangular decomposition $A = B \otimes N$ the A -module S_A is isomorphic to $\text{Ind}_N^A \text{Coind}_k^N \underline{k}$. The algebra A^\sharp plays the crucial part in all our considerations. In the Lie algebra case the semiregular representation appeared first in the paper of Voronov [V], the algebra A^\sharp in the general case was introduced in [Ar]. We compare the definition of semiinfinite Ext functor with the one from [Ar] and prove their coincidence.

The next section is devoted to the semiinfinite cohomology in the universal enveloping algebra case. Suppose we have a graded Lie algebra \mathfrak{g} and its triangular decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}$ as a graded vector space. Consider the universal enveloping algebra $A := U(\mathfrak{g})$ with the induced triangular decomposition. We prove that the algebra A^\sharp is also a universal enveloping algebra of a Lie algebra that differs from \mathfrak{g} by a one-dimensional central extension. This central extension is defined with the help of a 2-cocycle of \mathfrak{g} known as the *critical cocycle* (see e. g. [FFr]). Thus we prove that in the Lie algebra case our semiinfinite cohomology of the universal enveloping algebra coincides with Lie algebra semiinfinite cohomology (see e. g. [F]). We conclude the section with a direct calculation of the critical 2-cocycle for affine Lie algebras and for different triangular decompositions. It turns out that the cocycle itself depends on the type of the decomposition, still cocycles for different decompositions define the same cohomology class.

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1.3. **Conventions.** In most part of the paper we work over an arbitrary field k . All the vector spaces are considered over that field. In the end of the fourth section our base field is \mathbf{C} .

2. RELATIVE BAR CONSTRUCTION

2.1. Suppose we have a graded associative algebra $A = \bigoplus_{n \in \mathbf{Z}} A_n$. Let B and N be graded subalgebras in A satisfying the following conditions:

- (i) N is positively graded;
- (ii) $N_0 = k$;
- (iii) $\dim N_n < \infty$ for any $n \in \mathbf{N}$;
- (iv) B is negatively graded;
- (v) the multiplication in A defines the isomorphisms of graded vector spaces

$$B \otimes N \longrightarrow A \text{ and } N \otimes B \longrightarrow A.$$

In particular N is naturally augmented. We denote the augmentation ideal $\bigoplus_{n>0} N_n$ by \overline{N} .

The category of left graded A -modules with morphisms that preserve gradings is denoted by $A\text{-mod}$. We define the functor of the grading shift

$$A\text{-mod} \longrightarrow A\text{-mod} : M \mapsto M\langle i \rangle, \quad M\langle i \rangle_m := M_{i+m}, i \in \mathbf{Z}.$$

The space $\bigoplus_{i \in \mathbf{Z}} \text{Hom}_{A\text{-mod}}(M_1, M_2\langle i \rangle)$ is denoted by $\text{Hom}_A(M_1, M_2)$.

2.2. We fix a left B -augmentation on A provided by the isomorphism of graded left B -modules $B \cong A \otimes_N \underline{k}$ where $\underline{k} := N/\overline{N}$ is the trivial N -module. The A -module $A \otimes_N k$ is denoted by \underline{B} .

2.3. We introduce certain subcategories in the category of complexes $\text{Kom}(A\text{-mod})$. For $M^\bullet \in \text{Kom}(A\text{-mod})$ the support of M is defined as follows:

$$\text{supp } M^\bullet := \{(p, q) \in \mathbf{Z}^{\oplus 2} \mid M_p^q \neq 0\}.$$

For $s_1, s_2, t_1, t_2 \in \mathbf{Z}$, $s_1, s_2 > 0$, the set $\{(p, q) \in \mathbf{Z}^{\oplus 2} \mid s_1 q + p \geq t_1, s_2 q - p \geq t_2\}$ (resp. the set $\{(p, q) \in \mathbf{Z}^{\oplus 2} \mid s_1 q + p \leq t_1, s_2 q - p \leq t_2\}$) is denoted by $X^\uparrow(s_1, s_2, t_1, t_2)$ (resp. by $X^\downarrow(s_1, s_2, t_1, t_2)$).

2.3.1. Let $\mathcal{C}^\uparrow(A)$ (resp. $\mathcal{C}^\downarrow(A)$) be the full subcategory in $\text{Kom}(A\text{-mod})$ consisting of complexes M^\bullet that satisfy the following condition:

- (U) there exist $s_1, s_2, t_1, t_2 \in \mathbf{Z}$, $s_1, s_2 > 0$, such that $\text{supp } M^\bullet \subset X^\uparrow(s_1, s_2, t_1, t_2)$ (resp.
- (D) there exist $s_1, s_2, t_1, t_2 \in \mathbf{Z}$, $s_1, s_2 > 0$, such that $\text{supp } M^\bullet \subset X^\downarrow(s_1, s_2, t_1, t_2)$).

2.3.2. Recall the construction of standard relative bar resolution of a complex of A -modules M^\bullet .

The standard bar resolution $\widetilde{\text{Bar}}^\bullet(A, B, M) \in \text{Kom}(A\text{-mod})$ of an A -module M is defined as follows:

$$\begin{aligned} \widetilde{\text{Bar}}^{-n}(A, B, M) &:= A \otimes_B \dots \otimes_B A \otimes_B M \text{ (} n+1 \text{ times),} \\ d(a_0 \otimes \dots \otimes a_n \otimes v) &= \sum_{s=0}^{n-1} (-1)^s a_0 \otimes \dots \otimes a_s a_{s+1} \otimes \dots \otimes v + (-1)^n a_0 \otimes \dots \otimes a_{n-1} \otimes a_n v. \end{aligned}$$

Here $a_0, \dots, a_n \in A$, $v \in M$.

2.3.3. **Lemma:** The subspace $\overline{\text{Bar}}^\bullet(A, B, M)$:

$$(\overline{\text{Bar}})^{-n}(A, B, M) := \{a_0 \otimes \dots \otimes a_n \otimes v \in \widetilde{\text{Bar}}^{-n}(A, B, M) \mid \exists s \in \{1, \dots, n\} : a_s \in B\}$$

is a subcomplex in $\widetilde{\text{Bar}}^\bullet(A, B, M)$. \square

The quotient $\text{Bar}^\bullet(A, B, M) := \widetilde{\text{Bar}}^\bullet(A, B, M) / \overline{\text{Bar}}^\bullet(A, B, M)$ is called the restricted bar resolution of the A -module M with respect to the subalgebra B .

For a complex of A -modules $M^\bullet \in \mathcal{C}^\uparrow(A)$ its relative restricted bar resolution is defined as a total complex of the bicomplex $\text{Bar}^\bullet(A, B, M^\bullet)$.

2.3.4. **Lemma:** Let $M \in \mathcal{C}^\uparrow(A)$. Then

(i) $\text{Bar}^{-m}(A, B, M^\bullet) = N \otimes \overline{N}^{\otimes m} \otimes M^\bullet$ as a vector space;

(ii) $\text{Bar}^\bullet(A, B, M^\bullet) \in \mathcal{C}^\uparrow(A)$;

(iii) $\text{Bar}^\bullet(A, B, M^\bullet)$ is quasiisomorphic to M as a complex of A -modules. \square

Note that for $M \in A\text{-mod}$ the complex $\text{Bar}^\bullet(A, B, M)$ is isomorphic to $\text{Bar}^\bullet(N, k, M)$ as a complex of N -modules. Here k is considered as a subalgebra in N .

2.4. Consider the relative restricted bar resolution $\text{Bar}^\bullet(A, B, \underline{B})$ of the A -module \underline{B} and the complex of graded vector spaces

$$D^\bullet(A, B) := \text{Hom}_A^\bullet(\text{Bar}^\bullet(A, B, \underline{B}), \underline{B}).$$

Clearly $D^\bullet(A, B) \cong \text{Hom}_k^\bullet(\bigoplus_{n>0} \overline{N}^{\otimes n}, \underline{B})$ as a vector space, and $D(A, B)$ belongs to $\mathcal{C}^\downarrow(\text{Vect})$.

2.4.1. We introduce a structure of a DG-algebra on $D^\bullet(A, B)$. First we define a DG-algebra structure on $\text{Hom}_A^\bullet(\widetilde{\text{Bar}}^\bullet(A, B, \underline{B}), \underline{B})$. Note that by Schapiro lemma

$$\text{Hom}_A^\bullet(\widetilde{\text{Bar}}^{-m}(A, B, \underline{B}), \underline{B}) = \text{Hom}_B^\bullet(A \otimes_B \dots \otimes_B A \otimes_B \underline{B}, \underline{B}) \text{ (} m \text{ times)}.$$

Let $f \in \text{Hom}_A^m(\widetilde{\text{Bar}}^\bullet(A, B, \underline{B}), \underline{B})$, $g \in \text{Hom}_A^n(\widetilde{\text{Bar}}^\bullet(A, B, \underline{B}), \underline{B})$, i. e.

$$f : \underbrace{A \otimes_B \dots \otimes_B A}_{m} \otimes_B \underline{B} \longrightarrow \underline{B}, \quad g : \underbrace{A \otimes_B \dots \otimes_B A}_{n} \otimes_B \underline{B} \longrightarrow \underline{B},$$

both f and g are B -linear. By definition set $f \cdot g : \underbrace{A \otimes_B \dots \otimes_B A}_{m+n} \otimes_B \underline{B} \longrightarrow \underline{B}$:

$$(f \cdot g)(a_1 \otimes \dots \otimes a_{m+n} \otimes b) := f(a_1 \otimes \dots \otimes a_n \otimes g(a_{n+1} \otimes \dots \otimes a_{m+n} \otimes b)).$$

2.4.2. **Lemma:**

(i) The multiplication equips $\text{Hom}_A^\bullet(\widetilde{\text{Bar}}^\bullet(A, B, \underline{B}), \underline{B})$ with a DG-algebra structure;

(ii) $D^\bullet(A, B) = \{f \in \text{Hom}_A^\bullet(\widetilde{\text{Bar}}^\bullet(A, B, \underline{B}), \underline{B}) \mid f \equiv 0 \text{ on } \overline{\text{Bar}}^\bullet(A, B, \underline{B}) \subset \widetilde{\text{Bar}}^\bullet(A, B, \underline{B})\}$

is a DG-subalgebra in $\text{Hom}_A^\bullet(\widetilde{\text{Bar}}^\bullet(A, B, \underline{B}), \underline{B})$. \square

2.4.3. **Remark:** (i) $D^0(A, B) = \text{Hom}_A(A \otimes_B \underline{B}, \underline{B}) = \text{Hom}_B(\underline{B}, \underline{B}) \cong B^{\text{opp}}$ as an algebra (yet the inclusion $B^{\text{opp}} \hookrightarrow D^\bullet(A, B)$ is not a morphism of DG-algebras — the differential in $D^\bullet(A, B)$ does not preserve B^{opp}).

(ii) Consider the induction functor $\text{Ind}_N^A : N\text{-mod} \longrightarrow A\text{-mod}$. Then the canonical map

$$\begin{aligned} \text{Hom}_N^\bullet(\text{Bar}^\bullet(N, k, \underline{k}), \underline{k}) &\longrightarrow \text{Hom}_A^\bullet(\text{Ind}_N^A(\text{Bar}^\bullet(N, k, \underline{k})), \text{Ind}_N^A(\underline{k})) \\ &\cong \text{Hom}_A^\bullet(\text{Bar}^\bullet(A, B, \underline{B}), \underline{B}) = D^\bullet(A, B) \end{aligned}$$

is an inclusion of DG-algebras.

(iii) Denote its image by $D^\bullet(N, k)$. Then $D^\bullet(A, B) = D^\bullet(N, k) \otimes B^{\text{opp}}$ as a graded vector space.

2.4.4. Let $M^\bullet \in \mathcal{C}^\downarrow(A)$, $M'^\bullet \in \mathcal{C}^\uparrow(A^{\text{opp}})$. By definition set

$$D^\downarrow(A, B, M^\bullet) := \text{Hom}_A^\bullet(\text{Bar}^\bullet(A, B, \underline{B}), M^\bullet), \quad D^\uparrow(A, B, M'^\bullet) := M'^\bullet \otimes_A \text{Bar}^\bullet(A, B, \underline{B}).$$

Evidently the vector space $D^\downarrow(A, B, M^\bullet)$ (resp. $D^\uparrow(A, B, M'^\bullet)$) belongs to $\mathcal{C}^\downarrow(\text{Vect})$ (resp. to $\mathcal{C}^\uparrow(\text{Vect})$). Similarly to 2.4.1 we define the right action of $D^\bullet(A, B)$ on $D^\downarrow(A, B, M^\bullet)$ (resp the left action of $D^\bullet(A, B)$ on $D^\uparrow(A, B, M'^\bullet)$) as follows.

Recall that $\text{Hom}_A^\bullet(\widetilde{\text{Bar}}^{-m}(A, B, \underline{B}), M^\bullet) \cong \text{Hom}_B^\bullet(\underbrace{A \otimes_B \dots \otimes_B A}_m \otimes_B \underline{B}, M^\bullet)$.

Suppose we have $f : \underbrace{A \otimes_B \dots \otimes_B A}_m \otimes_B \underline{B} \longrightarrow M^\bullet$, $g : \underbrace{A \otimes_B \dots \otimes_B A}_n \otimes_B \underline{B} \longrightarrow \underline{B}$, both f and g are B -linear. Then by definition set $(f \cdot g) : \underbrace{A \otimes_B \dots \otimes_B A}_{m+n} \otimes_B \underline{B} \longrightarrow M^\bullet :$

$$(f \cdot g)(a_1 \otimes \dots \otimes a_{m+n} \otimes b) := f(a_1 \otimes \dots \otimes a_m \otimes g(a_{m+1} \otimes \dots \otimes a_{m+n} \otimes b)).$$

Note also that $M'^\bullet \otimes_A \widetilde{\text{Bar}}^\bullet(A, B, \underline{B}) \cong \bigoplus_{n \leq 0} M' \otimes_B \underbrace{(A \otimes_B \dots \otimes_B A)}_{-n} \otimes_B \underline{B}$.

Suppose we have

$$x = m \otimes a_1 \otimes \dots \otimes a_n \otimes b \in M'^\bullet \otimes_B \underbrace{A \otimes_B \dots \otimes_B A}_n \otimes_B \underline{B}, \quad f : \underbrace{A \otimes_B \dots \otimes_B A}_m \otimes_B \underline{B} \longrightarrow \underline{B},$$

and f is B -linear. Then by definition set $f \cdot x \in M'^\bullet \otimes_B \underbrace{(A \otimes_B \dots \otimes_B A)}_{n-m} \otimes_B \underline{B} :$

$$f \cdot x := m \otimes a_1 \otimes \dots \otimes a_{n-m} \otimes f(a_{n-m+1} \otimes \dots \otimes a_n \otimes b).$$

2.4.5. **Lemma:**

- (i) The multiplication equips $\text{Hom}_A^\bullet(\widetilde{\text{Bar}}^\bullet(A, B, \underline{B}), M^\bullet)$ with a structure of the right DG-module over $D^\bullet(A, B)$;
- (ii) $D^\downarrow(A, B, M^\bullet) \subset \text{Hom}_A^\bullet(\widetilde{\text{Bar}}^\bullet(A, B, \underline{B}), M^\bullet)$ is a DG-submodule;
- (iii) the multiplication equips $M'^\bullet \otimes_A \widetilde{\text{Bar}}^\bullet(A, B, \underline{B})$ with a structure of the left DG-module over $D^\bullet(A, B)$;
- (iv) $D^\uparrow(A, B, M'^\bullet)$ is a quotient DG-module of $M'^\bullet \otimes_A \widetilde{\text{Bar}}^\bullet(A, B, \underline{B})$. □

2.4.6. Denote the category of right (resp. left) DG-modules

$$M^\bullet = \bigoplus_{p,q \in \mathbf{Z}} M_p^q, \quad d : M_p^q \longrightarrow M_p^{q+1},$$

over $D^\bullet(A, B)$, with morphisms being morphisms of DG-modules that preserve also the second grading, by $D^\bullet(A, B)$ -mod (resp. by $D^\bullet(A, B)^{\text{opp}}$ -mod). The subcategory in $D^\bullet(A, B)$ -mod (resp. in $D^\bullet(A, B)^{\text{opp}}$ -mod) that consists of DG-modules satisfying the condition (D) (resp. (U)) is denoted by $\mathcal{C}^\downarrow(D^\bullet(A, B))$ (resp. by $\mathcal{C}^\uparrow(D^\bullet(A, B)^{\text{opp}})$).

The localizations of $\mathcal{C}^\downarrow(A)$, $\mathcal{C}^\uparrow(A^{\text{opp}})$, $\mathcal{C}^\downarrow(D^\bullet(A, B))$, $\mathcal{C}^\uparrow(D^\bullet(A, B))$ and $\mathcal{C}^\uparrow(D^\bullet(A, B)^{\text{opp}})$ by the class of quasiisomorphisms are denoted by $\mathcal{D}^\downarrow(A)$, $\mathcal{D}^\uparrow(A^{\text{opp}})$, $\mathcal{D}^\downarrow(D^\bullet(A, B))$, and $\mathcal{D}^\uparrow(D^\bullet(A, B)^{\text{opp}})$ respectively.

By 2.4.4 D^\downarrow and D^\uparrow define the functors

$$D_A^\downarrow : \mathcal{C}^\downarrow(A) \longrightarrow \mathcal{C}^\downarrow(D^\bullet(A, B)) \text{ and } D_A^\uparrow : \mathcal{C}^\uparrow(A^{\text{opp}}) \longrightarrow \mathcal{C}^\uparrow(D^\bullet(A, B)^{\text{opp}}).$$

2.4.7. **Theorem:**

- (i) The functor D_A^\downarrow is well defined as a functor from $\mathcal{D}^\downarrow(A)$ to $\mathcal{D}^\downarrow(D^\bullet(A, B))$;
- (ii) $D_A^\downarrow : \mathcal{D}^\downarrow(A) \longrightarrow \mathcal{D}^\downarrow(D^\bullet(A, B))$ is an equivalence of triangulated categories;
- (iii) the functor D_A^\uparrow is well defined as a functor from $\mathcal{D}^\uparrow(A^{\text{opp}})$ to $\mathcal{D}^\uparrow(D^\bullet(A, B)^{\text{opp}})$;
- (iv) $D_A^\uparrow : \mathcal{D}^\uparrow(A^{\text{opp}}) \longrightarrow \mathcal{D}^\uparrow(D^\bullet(A, B)^{\text{opp}})$ is an equivalence of triangulated categories.

2.5. **Proof of Theorem 2.4.7.** We present here the proof of the first two statements, the other two are verified in a similar way.

First we prove that D_A^\downarrow maps quasiisomorphisms into quasiisomorphisms.

2.5.1. **Lemma:** Let $M^\bullet \in \mathcal{C}^\downarrow(A)$ be exact. Then $D_A^\downarrow(M^\bullet) \in \mathcal{C}^\downarrow(D^\bullet(A, B))$ is also exact.

Proof. As a complex of vector spaces $D_A^\downarrow(M^\bullet) = \text{Hom}_N^\bullet(\text{Bar}^\bullet(N, k, \underline{k}), M^\bullet)$. Consider the spectral sequence of this bicomplex.

$$E_0^{p,q} = \text{Hom}_k((\overline{N})^{\otimes -p}, M^q) = \bigoplus_{n \in \mathbf{Z}} \bigoplus_{\sum t_j - s = n} \text{Hom}_k(N_{t_1} \otimes \dots \otimes N_{t_p}, M_s^q)$$

As $D_A^\downarrow(M^\bullet)$ satisfies the condition (D), in a fixed grading the spectral sequence is situated in an area of the (p, q) plane bounded both in p and q . Thus the spectral sequence converges. But the first term of it looks as follows.

$$E_1^{p,q} = \text{Hom}_k((\overline{N})^{\otimes -p}, H^q(M)).$$

In particular $E_1^{p,q} = 0$ if M^\bullet is exact. □

The first statement of the theorem is proved. To prove the second one we will construct a functor $\overline{D}_A^\downarrow : \mathcal{C}^\downarrow(D^\bullet(A, B)) \longrightarrow \mathcal{C}^\downarrow(A)$.

Consider the complex $K^\bullet := \text{Hom}_A^\bullet(\text{Bar}^\bullet(A, B, \underline{A}), \underline{B})$. As a vector space

$$K^\bullet = \text{Hom}_k^\bullet\left(\bigoplus_{n \in \mathbf{Z}} N \otimes \overline{N}^{\otimes n}, \underline{B}\right).$$

In particular $K^\bullet \in \mathcal{C}^\downarrow(\mathcal{V}ect)$. We define the *left* action of $D^\bullet(A, B)$ on K^\bullet similarly to 2.4.1. The left action of A on K^\bullet is defined using the right multiplication in the A - A bimodule \underline{A} .

2.5.2. **Lemma:** K^\bullet is a $D^\bullet(A, B)$ - A DG-bimodule. \square

Note that as a $D^\bullet(A, B)$ -module $K^\bullet \cong D^\bullet(A, B) \otimes_N N^*$ (the differentials both in $D^\bullet(A, B)$ and K^\bullet are forgotten).

Consider a functor $\overline{D}_A^\downarrow : D^\bullet(A, B)\text{-mod} \longrightarrow A\text{-mod}$, $X^\bullet \mapsto K^\bullet \otimes_{D^\bullet(A, B)} X^\bullet$.

2.5.3. **Lemma:**

- (i) $\overline{D}_A^\downarrow : \mathcal{C}^\downarrow(D^\bullet(A, B)) \longrightarrow \mathcal{C}^\downarrow(A)$;
- (ii) $\overline{D}_A^\downarrow$ is well defined on the corresponding derived categories.

Proof. (i) Note that $\overline{D}_A^\downarrow(X^\bullet) = N^* \otimes X^\bullet$ as a vector space.

(ii) The proof is parallel to the one of 2.5.1. \square

Note that for $M \in A\text{-mod}$ the complex $\text{Hom}_A^\bullet(\text{Bar}^\bullet(A, B, \underline{A}), M)$ is a complex of left A -modules. The multiplication by elements of A is provided by the right multiplication in the A -bimodule $\text{Bar}^\bullet(A, B, \underline{A})$.

Note also that the complex $\text{Hom}_A^\bullet(\text{Bar}^\bullet(A, B, \text{Bar}^\bullet(A, B, \underline{B})), \underline{B})$ is a DG-bimodule over $D^\bullet(A, B)$. The construction of the right and the left action of $D^\bullet(A, B)$ on the complex is as follows.

$$\text{Bar}^{-p}(A, B, \text{Bar}^{-q}(A, B, \underline{B})) = A \otimes_B \left(\underbrace{(A/B) \otimes_B \dots \otimes_B (A/B)}_p \right) \otimes_B A \otimes_B \left(\underbrace{(A/B) \otimes_B \dots \otimes_B (A/B)}_q \right) \otimes_B \underline{B}.$$

Suppose we have

$$\begin{aligned} f_1 &\in \text{Hom}_A^\bullet(\text{Bar}^{-s}(A, B, \underline{B}), \underline{B}) = \text{Hom}_B^\bullet(\underbrace{A \otimes_B \dots \otimes_B A}_{s} \otimes_B \underline{B}, \underline{B}), \\ f_2 &\in \text{Hom}_A^\bullet(\text{Bar}^{-t}(A, B, \underline{B}), \underline{B}) = \text{Hom}_B^\bullet(\underbrace{A \otimes_B \dots \otimes_B A}_{t} \otimes_B \underline{B}, \underline{B}), \\ g &\in \text{Hom}_A^\bullet(\text{Bar}^{-p}(A, B, \text{Bar}^{-q}(A, B, \underline{B})), \underline{B}) \hookrightarrow \text{Hom}_B^\bullet(\underbrace{A \otimes_B \dots \otimes_B A}_{p+q+1} \otimes_B \underline{B}, \underline{B}). \end{aligned}$$

Then set

$$\begin{aligned} (f_1 \cdot g \cdot f_2)(a_1 \otimes \dots \otimes a_{s+t+p+q+1} \otimes b) = \\ f_1(a_1 \otimes \dots \otimes a_s \otimes g(a_{s+1} \otimes \dots \otimes a_{s+p+q+1} \otimes f_2(a_{s+p+q+2} \otimes \dots \otimes a_{s+t+p+q+1} \otimes b))). \end{aligned}$$

One can check directly that the differential in the complex satisfies the Leibnitz rule both for the left and the right DG-module structures.

Consider a subspace $L^\bullet \subset \text{Bar}^\bullet(A, B, \text{Bar}^\bullet(A, B, \underline{B}))$ defined as follows.

$$\begin{aligned} L^{-p, -q} &:= A \otimes_B \left(\underbrace{(A/B) \otimes_B \dots \otimes_B (A/B)}_p \right) \otimes_B B \otimes_B \left(\underbrace{(A/B) \otimes_B \dots \otimes_B (A/B)}_q \right) \otimes_B \underline{B}, \\ L^\bullet &:= \bigoplus_{p, q \in \mathbb{Z}} L^{p, q} \hookrightarrow \text{Bar}^\bullet(A, B, \text{Bar}^\bullet(A, B, \underline{B})). \end{aligned}$$

2.5.4. **Lemma:** L^\bullet is a subcomplex in $\text{Bar}^\bullet(A, B, \text{Bar}^\bullet(A, B, \underline{B}))$. \square

2.5.5. Lemma:

(i) Let $M^\bullet \in \mathcal{C}^\downarrow(A)$. Then $\overline{D}_A^\downarrow \circ D_A^\downarrow(M^\bullet) \cong \text{Hom}_A^\bullet(\text{Bar}^\bullet(A, B, \underline{A}), M^\bullet)$ as a complex of A -modules.

(ii) Let $X^\bullet \in \mathcal{C}^\downarrow(D^\bullet(A, B))$. Then

$$D_A^\downarrow \circ \overline{D}_A^\downarrow(X^\bullet) \cong X^\bullet \otimes_{D^\bullet(A, B)} \text{Hom}_A^\bullet(\text{Bar}^\bullet(A, B, \text{Bar}^\bullet(A, B, \underline{B})), \underline{B}). \quad \square$$

The canonical morphism of A -modules $\text{Bar}^\bullet(A, B, \underline{A}) \rightarrow \underline{A}$ provides a morphism of functors $\phi : \text{Id}_{\mathcal{C}^\downarrow(A)} \rightarrow \overline{D}_A^\downarrow \circ D_A^\downarrow$.

The canonical morphism of complexes $\text{Bar}^\bullet(A, B, \underline{B}) \rightarrow \text{Bar}^\bullet(A, B, \text{Bar}^\bullet(A, B, \underline{B}))$:

$$\begin{aligned} \text{Bar}^{-m}(A, B, \underline{B}) &\rightarrow \bigoplus_{p+q=m} L^{-p, -q} \subset \text{Bar}^\bullet(A, B, \text{Bar}^\bullet(A, B, \underline{B})) \\ a_1 \otimes \dots \otimes a_{m+1} \otimes b &\mapsto \sum_{i=1}^{m+1} a_1 \otimes \dots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \dots \otimes a_{m+1} \otimes b \end{aligned}$$

provides a morphism of DG-bimodules over $D^\bullet(A, B)$

$$\text{Hom}_A^\bullet(\text{Bar}^\bullet(A, B, \text{Bar}^\bullet(A, B, \underline{B})), \underline{B}) \rightarrow D^\bullet(A, B)$$

that leads to the morphism of functors $\psi : D_A^\downarrow \circ \overline{D}_A^\downarrow \rightarrow \text{Id}_{\mathcal{C}^\downarrow(D^\bullet(A, B))}$.

2.5.6. Lemma: Both ϕ and ψ are quasiisomorphisms. \square

Thus Theorem 2.4.7 is proved. \square

2.5.7. Remark: To construct an inverse functor for D_A^\uparrow one has to consider a functor

$$\overline{D}_A^\uparrow : \mathcal{C}^\uparrow(D^\bullet(A, B)^{\text{opp}}) \rightarrow \mathcal{C}^\uparrow(A^{\text{opp}}), \quad \overline{D}_A^\uparrow : X^\bullet \mapsto \text{Hom}_{D^\bullet(A, B)^{\text{opp}}}^\bullet(K^\bullet, X^\bullet).$$

3. THE ALGEBRA A^\sharp AND SEMIINFINITE COHOMOLOGY

In this section we give a realization of semiinfinite cohomology in terms of bar duality. The classical construction of that type was obtained in [BGG]. The Koszul duality functor K constructed there gave an equivalence of suitably chosen derived categories of modules over a quadratic Koszul algebra A and its quadratic dual algebra A^\sharp (see [BGG], [BGS]). The counterpart of the classical Ext functor in that construction was given by the following statement.

3.1. Proposition: Let M be a graded module over a quadratic Koszul algebra A . Then $H^\bullet(K(M)) = \bigoplus_{n \in \mathbb{Z}} \text{Ext}_{A\text{-mod}}^\bullet(\underline{k}, M\langle n \rangle)$. Here as before \underline{k} denotes the trivial module over A placed in the zero grading and $\langle * \rangle$ denotes the shift in the category of graded modules. \square

For a graded algebra A and graded A -modules M_1 and M_2 we denote the space $\bigoplus_{n \in \mathbb{Z}} \text{Ext}_{A\text{-mod}}^\bullet(M_1, M_2\langle n \rangle)$ by $\text{Ext}_A^\bullet(M_1, M_2)$.

3.2. We return to the situation of the previous section.

Let $\widetilde{\mathcal{C}}^\downarrow(D^\bullet(A, B))$ be the category of DG-modules over $D^\bullet(A, B)$ whose cohomology satisfies the condition (D), $\widetilde{\mathcal{D}}^\downarrow(D^\bullet(A, B))$ denotes its localization by the class of quasiisomorphisms.

3.2.1. **Lemma:** The natural inclusion functor provides the equivalence of categories

$$\mathcal{D}^\downarrow(D^\bullet(A, B)) \xrightarrow{\sim} \widetilde{\mathcal{D}}^\downarrow(D^\bullet(A, B)).$$

Proof. We construct truncation functors similar to those described in [GM] for the usual categories of complexes over abelian categories.

First note that for any $M^\bullet \in D^\bullet(A, B)\text{-mod}$ and an arbitrary integer n the complex

$$\sigma_{\leq n} M^\bullet : \sigma_{\leq n} M_p^q := \begin{cases} M_p^q, & \text{if } p \leq n, \\ 0, & \text{if } p > n \end{cases}$$

is a DG-submodule in M^\bullet .

Denote the set $\{(p, q) \in \mathbf{Z}^{\oplus 2} \setminus X^\downarrow(s_1, s_2, t_1, t_2) | (p+1, q) \in X^\downarrow(s_1, s_2, t_1, t_2)\}$. (resp. the set $\{(p, q) \in \mathbf{Z}^{\oplus 2} \setminus X^\downarrow(s_1, s_2, t_1, t_2) | (p-1, q) \in X^\downarrow(s_1, s_2, t_1, t_2)\}$). by $X^\swarrow(s_1, s_2, t_1, t_2)$ (resp. by $X^\searrow(s_1, s_2, t_1, t_2)$).

Choose s_1, s_2, t_1, t_2 such that

$$\text{supp } H^\bullet(M^\bullet) \subset X^\downarrow(s_1, s_2, t_1, t_2) \text{ and } \text{supp } D^\bullet(A, B) \subset X^\downarrow(s_1, s_2, 0, 0).$$

Let (p_0, q_0) be the vertex of the convex cone $X^\downarrow(s_1, s_2, t_1, t_2)$. Consider a submodule $\tau_{\swarrow}(s_1, s_2, t_1, t_2)(M^\bullet)$ in $\sigma_{\leq p_0} M^\bullet$:

$$\tau_{\swarrow}(s_1, s_2, t_1, t_2)(M^\bullet)_p^q := \begin{cases} 0, & \text{if } p > p_0 \\ \text{Ker}(d_p^q), & \text{if } (p, q) \in X^\swarrow(s_1, s_2, t_1, t_2) \\ M_p^q, & \text{otherwise.} \end{cases}$$

One can check directly that the defined submodule is quasiisomorphic to M^\bullet .

Consider also a submodule $\tau_{\searrow}(s_1, s_2, t_1, t_2)(M^\bullet)$ in $\sigma_{\leq p_0} M^\bullet$:

$$\tau_{\searrow}(s_1, s_2, t_1, t_2)(M^\bullet)_p^q := \begin{cases} 0, & \text{if } p > p_0 \\ \text{Im}(d_p^q), & \text{if } (p, q) \in X^\searrow(s_1, s_2, t_1, t_2) \\ M_p^q, & \text{otherwise.} \end{cases}$$

One can check directly that the defined submodule is quasiisomorphic to M^\bullet .

But $\tau_{\searrow}(s_1, s_2, t_1, t_2)\tau_{\swarrow}(s_1, s_2, t_1, t_2)M^\bullet$ satisfies the condition (D). Thus every DG-module from $\widetilde{\mathcal{C}}^\downarrow(D^\bullet(A, B))$ is quasiisomorphic to a DG-module from $\mathcal{C}^\downarrow(D^\bullet(A, B))$.

That means that the inclusion functor $\mathcal{D}^\downarrow(D^\bullet(A, B)) \longrightarrow \widetilde{\mathcal{D}}^\downarrow(D^\bullet(A, B))$ is surjective on classes of isomorphism of objects. On the other hand let

$$M_1^\bullet, M_3^\bullet \in \mathcal{C}^\downarrow(D^\bullet(A, B)), M_2^\bullet \in \widetilde{\mathcal{C}}^\downarrow(D^\bullet(A, B)), M_1^\bullet \xleftarrow{s} M_2^\bullet \xrightarrow{f} M_3^\bullet,$$

where s is a quasiisomorphism, be a diagram in $\widetilde{\mathcal{C}}^\downarrow(D^\bullet(A, B))$ representing a morphism in $\widetilde{\mathcal{D}}^\downarrow(D^\bullet(A, B))$ (see [GM] for the explicit description of morphisms in derived categories). Then there exist s_1, s_2, t_1, t_2 such that the diagram is equivalent to the following one:

$$M_1^\bullet \longleftarrow \tau_{\swarrow}(s_1, s_2, t_1, t_2)\tau_{\searrow}(s_1, s_2, t_1, t_2)M_2^\bullet \longrightarrow M_3^\bullet.$$

That means that the inclusion functor on derived categories is surjective on the class of morphisms. In the same way one checks that if two diagrams represent the same morphism in $\widetilde{\mathcal{D}}^\downarrow(D^\bullet(A, B))$ then the corresponding truncated diagrams are also equivalent in $\mathcal{D}^\downarrow(D^\bullet(A, B))$. Thus the inclusion functor is an equivalence of categories. \square

3.2.2. Lemma: Let $P^{\bullet,\bullet}$ be a bicomplex over an abelian category \mathcal{A} such that for $r \gg 0$ all $P^{r,\bullet} = 0$ and for every $r \in \mathbf{Z}$ the complex $P^{r,\bullet}$ is homotopic to zero. Then the total complex P^\bullet is also homotopic to zero.

Proof. The differential in the bicomplex $P^{\bullet,\bullet}$ is the sum of two components of the bigradings $(0, 1)$ and $(1, 0)$ respectively: $d = d_1 + d_2$. Choose a homotopy map

$$h_r : P^{r,\bullet} \longrightarrow P^{r,\bullet-1}; \quad d_1 h_r + h_r d_1 = \text{Id}_{P^{r,\bullet}}.$$

Set $h := \sum h_r$. Then the map $dh + hd = \text{Id}_{P^{\bullet,\bullet}} + d_2 h + h d_2$ is the sum of an invertible map and a nilpotent one. Thus it is invertible itself. \square

We say that two morphisms of DG-modules over $D^\bullet(A, B)$ are homotopic to each other: $f_1 \sim f_2$, $f_1, f_2 : M_{(1)}^\bullet \longrightarrow M_{(2)}^\bullet$, if there exists a morphism of bigraded $D^\bullet(A, B)$ -modules

$$h : M_{(1)}^\bullet \longrightarrow M_{(2)}^\bullet, \quad h_p^q : M_{(1)p}^q \longrightarrow M_{(2)p}^{q-1},$$

such that $f_1 - f_2 = dh + hd$. Two DG-modules M_1 and M_2 are homotopically equivalent, $M_1 \sim M_2$, if there exist morphisms of DG-modules

$$f : M_1 \longrightarrow M_2 \text{ and } g : M_2 \longrightarrow M_1 : fg \sim \text{Id}_{M_2}, \quad gf \sim \text{Id}_{M_1}.$$

A DG-module M^\bullet is homotopic to zero if $\text{Id}_M \sim 0$.

Now we are to construct a realization of the category $\mathcal{D}^\downarrow(D^\bullet(A, B))$ similar to the realization of the usual bounded derived category of an abelian category as the homotopical category of complexes consisting of projective objects.

3.2.3. Lemma: Let $M^\bullet \in D^\bullet(A, B)\text{-mod}$ be an exact DG-module. Then the DG-module $\text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet)$ is homotopic to zero.

Proof. Consider a bicomplex $\text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet)$:

$$\text{Bar}^{u,v}(D^\bullet(A, B), k, M^\bullet) := \bigoplus_r \bigoplus_{\substack{u=-n, q+\sum q_i+t=v \\ p_i \neq 0, p+\sum p_i+s=r}} D_p^q(A, B) \otimes D_{p_1}^{q_1}(A, B) \otimes \dots \otimes D_{p_n}^{q_n}(A, B) \otimes M_s^t.$$

The differential in the bicomplex is a sum of two components d_1 and d_2 . The bicomplex $\text{Bar}^{u,v}(D^\bullet(A, B), k, M^\bullet)$ is bounded from above in u .

Note that each of the complexes of vector spaces $\overline{D}^\bullet(A, B)^{\otimes p} \otimes M^\bullet$ is exact. Choose homotopy maps h_p for these complexes. Then $\text{Id} \otimes h_u$ will be a homotopy map in the $D^\bullet(A, B)$ -module $D^\bullet(A, B) \otimes \overline{D}^\bullet(A, B)^{\otimes u} \otimes M^\bullet$ with the component of the differential d_2 forgotten. By Lemma 3.2.2 the DG-module $\text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet)$ is itself homotopic to zero. \square

3.2.4. Corollary: For $M^\bullet \in \mathcal{C}^\downarrow(D^\bullet(A, B))$

$$\text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet) \sim \text{Bar}^\bullet(D^\bullet(A, B), k, \text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet)). \quad \square$$

The homotopical category $\mathcal{K}(\mathcal{C}^\downarrow(D^\bullet(A, B)))$ is the category with the class of objects the same as in $\mathcal{C}^\downarrow(D^\bullet(A, B))$ and morphisms being classes of homotopical equivalence of morphisms in $\mathcal{C}^\downarrow(D^\bullet(A, B))$. Standard considerations (see [GM]) show that it is a triangulated category.

Consider the full subcategory $\mathcal{BAR}(D^\bullet(A, B))$ in the category $\mathcal{K}(D^\bullet(A, B)\text{-mod})$, a DG-module $M^\bullet \in \mathcal{BAR}(D^\bullet(A, B))$ iff M^\bullet is isomorphic to some DG-module of the type $\text{Bar}^\bullet(D^\bullet(A, B), k, M')$. Then Corollary 3.2.4 implies that $\mathcal{BAR}(D^\bullet(A, B))$ is a triangulated category: one has to check that for every morphism of DG-modules

$$f : \text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet) \longrightarrow \text{Bar}^\bullet(D^\bullet(A, B), k, M'^\bullet)$$

the DG-module $\text{Cone } f$ is also homotopically equivalent to a DG-module in $\mathcal{BAR}(D^\bullet(A, B))$. Now consider a morphism

$$\text{Bar}(f) : \text{Bar}^\bullet(D^\bullet(A, B), k, \text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet)) \longrightarrow \text{Bar}^\bullet(D^\bullet(A, B), k, \text{Bar}^\bullet(D^\bullet(A, B), k, M'^\bullet))$$

that corresponds to f . Since $\mathcal{K}(D^\bullet(A, B)\text{-mod})$ is a triangulated category itself, $\text{Cone}(f) \sim \text{Cone}(\text{Bar}(f))$. But $\text{Cone}(\text{Bar}(f)) \cong \text{Bar}^\bullet(D^\bullet(A, B), k, \text{Cone}(f))$ and we are done.

Let $\mathcal{D}(D^\bullet(A, B))$ be the localization of $D^\bullet(A, B)\text{-mod}$ by the class of quasiisomorphisms,

3.2.5. Proposition: The functor of localization by the class of quasiisomorphisms provides an equivalence of triangulated categories $L : \mathcal{BAR}(D^\bullet(A, B)) \xrightarrow{\sim} \mathcal{D}(D^\bullet(A, B))$.

Proof. Let us construct the inverse functor. Consider a functor

$$\bar{L} : D^\bullet(A, B)\text{-mod} \longrightarrow \mathcal{BAR}(D^\bullet(A, B)), \quad M^\bullet \mapsto \text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet).$$

By Lemma 3.2.3 the functor \bar{L} takes exact DG-modules to zero, thus quasiisomorphisms are mapped into quasiisomorphisms. By the characteristic property of the derived category \bar{L} induces a functor $\mathcal{D}(D^\bullet(A, B)) \longrightarrow \mathcal{BAR}(D^\bullet(A, B))$. This functor is the one we need. \square

3.2.6. Lemma: Let $M^\bullet, M'^\bullet \in D^\bullet(A, B)\text{-mod}$. Then

$$\text{Hom}_{\mathcal{K}(D^\bullet(A, B))}(\text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet), M'^\bullet) = \text{Hom}_{\mathcal{D}(D^\bullet(A, B))}(M^\bullet, M'^\bullet).$$

Proof. Since \bar{L} is an equivalence of categories,

$$\begin{aligned} & \text{Hom}_{\mathcal{D}(D^\bullet(A, B))}(M^\bullet, M'^\bullet) \\ &= \text{Hom}_{\mathcal{BAR}(D^\bullet(A, B))}(\text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet), \text{Bar}^\bullet(D^\bullet(A, B), k, M'^\bullet)) \\ &= H^0 \left(\text{Hom}_{D^\bullet(A, B)}^\bullet(\text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet), \text{Bar}^\bullet(D^\bullet(A, B), k, M'^\bullet)) \right)_0. \end{aligned}$$

Here $(*)_0$ denotes the zero grading component.

By the same arguments as in 3.2.3 it is proved that the functor

$$\begin{aligned} X^\bullet &\mapsto \text{Hom}_{D^\bullet(A, B)}^\bullet(\text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet), X^\bullet), \\ \text{Hom}_{D^\bullet(A, B)}^n(\text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet), X^\bullet) &= \prod_{q-p=n} \text{Hom}_{D^\bullet(A, B)}(\text{Bar}^p(D^\bullet(A, B), k, M^\bullet), X^q) \end{aligned}$$

maps exact DG-modules $X^\bullet \in D^\bullet(A, B)\text{-mod}$ into exact complexes of vector spaces.

Thus

$$\begin{aligned} & \text{Hom}_{\mathcal{BAR}(D^\bullet(A, B))}(\text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet), \text{Bar}^\bullet(D^\bullet(A, B), k, M'^\bullet)) \\ &= \text{Hom}_{\mathcal{BAR}(D^\bullet(A, B))}^\bullet(\text{Bar}^\bullet(D^\bullet(A, B), k, M^\bullet), M'^\bullet). \quad \square \end{aligned}$$

Recall that the complex $K^\bullet = \text{Hom}^\bullet(\text{Bar}^\bullet(A, B, A), \underline{B})$ is both a *left* DG-module over $D^\bullet(A, B)$ and a left A -module. Denote the category of left DG-modules over $D^\bullet(A, B)$ satisfying the condition (D) by $\mathcal{C}^\downarrow(D^\bullet(A, B)^{\text{opp}})$. Evidently $K^\bullet \in \mathcal{C}^\downarrow(D^\bullet(A, B)^{\text{opp}})$.

3.2.7. **Lemma:** The natural morphism of algebras $A \longrightarrow \text{Ext}_{D^\bullet(A,B)^{\text{opp}}}^0(K^\bullet, K^\bullet)$ is an inclusion.

Proof. Consider the bar resolution $\text{Bar}(D^\bullet(A, B), k, K^\bullet)$. By the previous Proposition

$$\text{Ext}_{D^\bullet(A,B)^{\text{opp}}}^0(K^\bullet, K^\bullet) = H^0 \left(\text{Hom}_{D^\bullet(A,B)^{\text{opp}}}(\text{Bar}^\bullet(D^\bullet(A, B), k, K^\bullet), \text{Bar}^\bullet(D^\bullet(A, B), k, K^\bullet)) \right).$$

Consider a functor

$$F : \mathcal{C}^\downarrow(D^\bullet(A, B)) \longrightarrow \text{Kom}(A\text{-mod}) : X^\bullet \mapsto X^\bullet \otimes_{D^\bullet(A, B)} \text{Bar}^\bullet(D^\bullet(A, B), k, K^\bullet)$$

One checks directly that F defines a functor on the corresponding derived categories that is equivalent to $\overline{D}_A^\downarrow$.

Suppose there exists $a \in A$ such that $a \mapsto 0$ in $\text{Ext}_{D^\bullet(A,B)^{\text{opp}}}^0(K^\bullet, K^\bullet)$. Then the multiplication by a defines an endomorphism of $\text{Bar}^\bullet(D^\bullet(A, B), k, K^\bullet)$ that is homotopic to zero, i. e. there exists a $D^\bullet(A, B)$ -linear map

$$h : \text{Bar}^\bullet(D^\bullet(A, B), k, K^\bullet) \longrightarrow \text{Bar}^{\bullet-1}(D^\bullet(A, B), k, K^\bullet)$$

(not commuting with the differential) such that $dh + hd = \mu_a$. Here μ_a denotes the endomorphism corresponding to $a \in A$.

Whence for every $X^\bullet \in \mathcal{C}^\downarrow(D^\bullet(A, B))$ the homomorphism of complexes $F(X^\bullet) \longrightarrow F(X^\bullet)$ given by the multiplication by a is homotopic to zero, in particular $a \in A$ acts trivially on the cohomology spaces $H^\bullet(F(X^\bullet))$.

The comparison of this fact with Theorem 2.4.7 finishes the proof of the Lemma. \square

3.3. Now we are going to introduce an algebra A^\sharp such that $A^\sharp = B^{\text{opp}} \otimes N^{\text{opp}}$ as a vector space and $D^\bullet(A, B)^{\text{opp}} \cong D^\bullet(A^\sharp, B^{\text{opp}})$.

Recall that the DG-algebra $D^\bullet(A, B)$ is a tensor product of the DG-subalgebra $D^\bullet(N, k)$ and the subalgebra (not a DG-subalgebra) B^{opp} . Recall also that the algebra $D^\bullet(N, k)$ is isomorphic to the tensor algebra over the graded vector space \overline{N}^* and the differential in it is generated by the map $\overline{N}^* \longrightarrow \overline{N}^* \otimes \overline{N}^*$ dual to the multiplication map in N and extended to the whole algebra $T(\overline{N}^*)$ by the Leibnitz rule.

To define a DG-algebra C^\vee such that C^\vee is a tensor product of its DG-subalgebra $T(\overline{N}^*)$ with a differential given by the map dual to the multiplication in N and a (not DG-) subalgebra B^{opp} one has to specify the following additional data:

- a linear map $B \otimes \overline{N}^* \longrightarrow \overline{N}^* \otimes B$ generating the multiplication in C^\vee ;
- a linear map $B \longrightarrow \overline{N}^* \otimes B$ providing a component of the differential in C^\vee ,

satisfying certain constraints (that provide the associativity constraint and the Leibnitz rule in the DG-algebra C^\vee).

On the other hand to define an algebra C such that C is a tensor product of two its subalgebras B and N as a vector space and the conditions 2.1 are satisfied one has to specify the following data (additional to the algebra structures on B and N):

- a linear map $\overline{N} \otimes B \longrightarrow B \oplus B \otimes \overline{N}$ providing the multiplication in C

satisfying certain constraints (that provide the associativity constraint in the algebra C).

3.3.1. Proposition: The construction of the dual algebra $C \mapsto C^\vee := D^\bullet(C, B)$ provides a one to one correspondence between the two described types of data, i. e. for every DG-algebra C^\vee such that $C^\vee = T(\overline{N}^*) \otimes B^{\text{opp}}$ as a vector space there exists an algebra $C = B \otimes N$ as a vector space such that the DG-algebras C^\vee and $D^\bullet(C, B)$ are isomorphic.

Proof. Evidently the two described types of data are dual to each other. One has to check directly that the constraints in the first and the second case are equivalent, i. e. that the data of the first type defines a DG-algebra (with the associativity constraint and the Leibnitz rule satisfied) if and only if the dual data of the second type defines an associative algebra. \square

3.3.2. Lemma: The DG-algebras $D^\bullet(N, k)^{\text{opp}}$ and $D^\bullet(N^{\text{opp}}, k)$ are isomorphic to each other. \square

3.3.3. Corollary: There exists an associative algebra A^\sharp such that A^\sharp contains two subalgebras B^{opp} and N^{opp} satisfying the conditions 2.1 for A^\sharp , B^{opp} and N^{opp} such that the DG-algebra $D^\bullet(A, B)^{\text{opp}} \cong D^\bullet(A^\sharp, B^{\text{opp}})$. \square

3.3.4. Corollary: The functor $D_{A^\sharp}^\perp$ provides an equivalence of triangulated categories $\mathcal{D}^\perp(A^\sharp) \xrightarrow{\sim} \mathcal{D}^\perp(D^\bullet(A, B)^{\text{opp}})$. \square

Now we describe the algebra A^\sharp explicitly.

3.4. Consider a left graded N -module $N^* := \bigoplus_{n \geq 0} \text{Hom}_k(N_n, k)$. The action of N on the space is defined as follows.

$$f : N \longrightarrow k, n \in N, \text{ then } (n \cdot f)(n') := f(n'n).$$

Consider also the left A -module $S_A := \text{Ind}_N^A(N^*) = A \otimes_N N^*$. Evidently $S_A \cong B \otimes N^*$ as a B -module and by 2.1 $S_A \cong N^* \otimes B$ as a N -module.

There is another left A -module with these two properties. $S'_A := \text{Hom}_B(A, B)$ with the left action of A defined as follows.

$$f : A \longrightarrow B, a \in A, \text{ then } (a \cdot f)(a') := f(a'a).$$

3.4.1. Lemma: The A -modules S_A and S'_A are isomorphic.

Proof. Fix the decomposition $A = B \otimes N$ provided by the multiplication in A . For $f \in N^*$ define $f_2 : A \longrightarrow B$, $f_2(b \otimes n) := f(n)b$. Define the pairing

$$S_A \times A \longrightarrow B, f \otimes a \times a' \mapsto f_2(a'a).$$

One checks directly the correctness of the definition and the perfectness of the defined pairing. \square

Thus $S_A \cong \text{Coind}_B^A B$. The functors of induction from N to A and of coinduction from B to A provide the natural inclusions of algebras

$$N^{\text{opp}} \hookrightarrow \text{Hom}_A(S_A, S_A) \text{ and } B^{\text{opp}} \hookrightarrow \text{Hom}_A(S_A, S_A).$$

3.4.2. Lemma: (i) $\text{Hom}_A(S_A, S_A) = \text{Hom}_k(N^*, B)$ as a vector space;

(ii) the subalgebra in $\text{Hom}_A(S_A, S_A)$ generated by the images of N^{opp} and B^{opp} is isomorphic to $B^{\text{opp}} \otimes N^{\text{opp}}$ as a vector space. \square

The latter algebra is denoted by A^\sharp .

3.4.3. Theorem: The algebras A^\sharp and A^\natural are isomorphic.

Proof. Consider the DG-module $D^\perp(S_A)$.

3.4.4. **Lemma:** The right $D^\bullet(A, B)$ DG-modules $D^\perp(S_A)$ and $\text{Ind}_{D^\bullet(N, k)}^{D^\bullet(A, B)} \underline{k}$ are quasiisomorphic to each other. \square

Consider also the left DG-module over $D^\bullet(A, B)^{\text{opp}}$, that is, the right DG $D^\bullet(A, B)$ -module $K_{A^\sharp}^\bullet$.

3.4.5. **Lemma:** $K_{A^\sharp}^\bullet$ is quasiisomorphic to $\text{Ind}_{D^\bullet(N, k)}^{D^\bullet(A, B)} \underline{k}$. \square

By Theorem 2.4.7 the algebra $\text{Hom}_A(S_A, S_A)$ is isomorphic to $\text{Ext}_{D^\bullet(A, B)}^0(D^\perp(S_A), D^\perp(S_A))$. Hence $A^\sharp \hookrightarrow \text{Ext}_{D^\bullet(A, B)}^0(\text{Ind}_{D^\bullet(N, k)}^{D^\bullet(A, B)} \underline{k}, \text{Ind}_{D^\bullet(N, k)}^{D^\bullet(A, B)} \underline{k})$.

By Lemma 3.2.7 $A^\sharp \hookrightarrow \text{Ext}_{D^\bullet(A, B)}^0(\text{Ind}_{D^\bullet(N, k)}^{D^\bullet(A, B)} \underline{k}, \text{Ind}_{D^\bullet(N, k)}^{D^\bullet(A, B)} \underline{k})$.

One can check directly that the images of the subalgebras B^{opp} (resp. N^{opp}) under these two inclusions coincide. \square

3.4.6. **Remark:** The previous Theorem provides an explicit way to recover the algebra $C = B \otimes N$ from $C^\vee = B^{\text{opp}} \otimes D^\bullet(N, k)$. Namely consider the *left* C^\vee DG-module $\text{Ind}_{D^\bullet(N, k)}^{C^\vee} \underline{k}$. Since $\text{Ext}_{D^\bullet(N, k)}^0(\underline{k}, \underline{k}) = N$ (this is a very easy case of Theorem 2.4.7), the algebra N acts on $\text{Ind}_{D^\bullet(N, k)}^{C^\vee} \underline{k}$ by endomorphisms. Since $D^\bullet(N, k)$ is normal in C^\vee , i. e. the left and the right ideals in C^\vee generated by its augmentation ideal coincide, the algebra B also acts on $\text{Ind}_{D^\bullet(N, k)}^{C^\vee} \underline{k}$ by endomorphisms.

Then the subalgebra in $\text{Ext}_{C^\vee}^0(\text{Ind}_{D^\bullet(N, k)}^{C^\vee} \underline{k}, \text{Ind}_{D^\bullet(N, k)}^{C^\vee} \underline{k})$ generated by the images of B and N is isomorphic to C .

3.4.7. **Lemma:** Let $C_{(1)}^\bullet$ and $C_{(2)}^\bullet$ be two DG-algebras,

$$C_{(1)}^\bullet = \bigoplus_{p, q \in \mathbf{Z}} C_{(1)p}^q, \quad C_{(2)}^\bullet = \bigoplus_{p, q \in \mathbf{Z}} C_{(2)p}^q,$$

and both $C_{(1)}^\bullet$ and $C_{(2)}^\bullet$ satisfy the condition (D). Let $\varphi : C_{(1)}^\bullet \rightarrow C_{(2)}^\bullet$ be a quasiisomorphism of DG-algebras. Then the restriction functor provided by φ is an equivalence of triangulated categories

$$\mathcal{D}^\perp(C_{(2)}^\bullet) \xrightarrow{\sim} \mathcal{D}^\perp(C_{(1)}^\bullet). \quad \square$$

In particular the algebra C can be recovered from the quasiisomorphism class of the DG-algebra C^\vee .

3.5. Consider a subcategory $\mathcal{C}^b(D^\bullet(A, B)) \hookrightarrow D^\bullet(A, B)\text{-mod}$ that consists of DG-modules satisfying both conditions (U) and (D). Its localization by the class of quasiisomorphisms is denoted by $\mathcal{D}^b(D^\bullet(A, B))$.

3.5.1. **Lemma:** Both inclusion functors

$$I^\uparrow : \mathcal{D}^b(D^\bullet(A, B)) \rightarrow \mathcal{D}^\uparrow(D^\bullet(A, B)) \text{ and } I^\downarrow : \mathcal{D}^b(D^\bullet(A, B)) \rightarrow \mathcal{D}^\downarrow(D^\bullet(A, B))$$

are full and faithful.

Proof. Like the statement of Lemma 3.2.1, the Lemma is an easy consequence of the existence of truncation functors. \square

Consider a full triangulated subcategory $\mathcal{D}(\text{Verma}^*(A))$ in $\mathcal{D}^\downarrow(A)$ (finitely) generated by all left A -modules of the form $\text{Coind}_B^A L_0$, where L_0 is some finite dimensional B -module, and a full triangulated subcategory $\mathcal{D}(\text{Verma}(A^{\sharp\text{opp}}))$ in $\mathcal{D}^\uparrow(A^{\sharp\text{opp}})$ generated by all right A^\sharp -modules of the form $\text{Ind}_B^{A^{\sharp\text{opp}}} L_0$, where L_0 is some finite dimensional B -module.

3.5.2. Proposition: The functor $\overline{D}_{A^\sharp}^\dagger \circ D_A^\downarrow$ provides an equivalence of triangulated categories $\mathcal{D}(\text{Verma}^*(A))$ and $\mathcal{D}(\text{Verma}(A^\sharp))$.

Proof. Note that for any left A -module of the form $M = \text{Coind}_B^A L_0$, where the B -module L_0 is finite dimensional, $D_A^\downarrow(M)$ belongs to $\mathcal{C}^b(D^\bullet(A, B))$ and for any right A^\sharp -module of the form $M' = \text{Ind}_B^{A^{\sharp\text{opp}}} L_0$, where the B -module L_0 is finite dimensional, $D_{A^\sharp}^\uparrow(M')$ belongs to $\mathcal{C}^b(D^\bullet(A^\sharp, B)^{\text{opp}}) \xrightarrow{\sim} \mathcal{C}^b(D^\bullet(A, B))$.

Now compare Theorem 2.4.7 with the previous Lemma. \square

3.5.3. Corollary: The additive category of left A -modules of the form $\text{Coind}_B^A L_0$ is equivalent to the additive category of right A^\sharp -modules of the form $\text{Ind}_B^{A^{\sharp\text{opp}}} L_0$, where L_0 is finite dimensional. \square

3.6. Semiinfinite cohomology of associative algebras. Here we give a definition of associative algebra semiinfinite cohomology and compare it with the one presented in [Ar].

3.6.1. Definition: Let $M^\bullet \in \mathcal{C}^\uparrow(A^{\text{opp}})$, $M'^\bullet \in \mathcal{C}^\downarrow(A^\sharp)$. Then set

$$\text{Ext}_A^{\frac{\infty}{2}+\bullet}(M'^\bullet, M^\bullet) := \text{Ext}_{D^\bullet(A, B)^{\text{opp}}}^\bullet(D_{A^\sharp}^\downarrow(M'^\bullet), D_A^\uparrow(M^\bullet)).$$

Note that by definition the semiinfinite Ext functor maps complexes exact by either of the variables to zero.

The definition of semiinfinite cohomology in [Ar] used the following standard complex.

$$C^{\frac{\infty}{2}+\bullet}(M'^\bullet, M^\bullet) := \text{Hom}_{A^\sharp}^\bullet(\text{Bar}^\bullet(A^\sharp, B^{\text{opp}}, M'^\bullet), \text{Bar}^\bullet(A^{\text{opp}}, N^{\text{opp}}, M^\bullet) \otimes_A S_A).$$

3.6.2. Theorem: Let $M^\bullet \in \mathcal{C}^\uparrow(A^{\text{opp}})$, $M'^\bullet \in \mathcal{C}^\downarrow(A^\sharp)$. Then $\text{Ext}_A^{\frac{\infty}{2}+\bullet}(M'^\bullet, M^\bullet) = H^\bullet(C^{\frac{\infty}{2}+\bullet}(M'^\bullet, M^\bullet))$.

Proof. First we present several Lemmas.

3.6.3. Lemma: $S_{A^\sharp} \cong S_A$ as an $A - A^\sharp$ bimodule. \square

3.6.4. Lemma: $\text{Bar}^\bullet(D^\bullet(A, B)^{\text{opp}}, D^\bullet(N^{\text{opp}}, k), D_{A^\sharp}^\downarrow(M'^\bullet)) \cong D_{A^\sharp}^\downarrow(\text{Bar}^\bullet(A^\sharp, N^{\text{opp}}, M'^\bullet))$ as a left $D^\bullet(A, B)$ DG-module. \square

3.6.5. Lemma: $K_{A^\sharp}^\bullet \cong D_A^\downarrow(S_A)$ as a $D^\bullet(A, B) - A^\sharp$ DG-bimodule. \square

Let us construct an explicit isomorphism of complexes

$$C^{\frac{\infty}{2}+\bullet}(M'^\bullet, M^\bullet) \cong \text{Hom}_{D^\bullet(A, B)^{\text{opp}}}^\bullet(\text{Bar}^\bullet(D^\bullet(A, B)^{\text{opp}}, D^\bullet(N^{\text{opp}}, k), D_{A^\sharp}^\downarrow(M'^\bullet)), D_A^\uparrow(M^\bullet)).$$

Using Lemma 3.6.4 and recalling that the functor $\overline{D}_{A^\sharp}^\downarrow$ is right adjoint to $D_{A^\sharp}^\downarrow$ we see that

$$RHS = \text{Hom}_{A^\sharp}^\bullet(\text{Bar}^\bullet(A^\sharp, N^{\text{opp}}, M'^\bullet), K_{A^\sharp}^\bullet \otimes_{D^\bullet(A, B)} D_A^\uparrow(M^\bullet)).$$

As before we use the left A^\sharp -module structure on $K_{A^\sharp}^\bullet$. Note also that as a graded module over $D^\bullet(A, B)$ the complex $D_A^\downarrow(S_A)$ is equal to $\text{Ind}_B^{D^\bullet(A, B)} S_A$. Thus using Lemma 3.6.5 we obtain that

$$K_{A^\sharp}^\bullet \otimes_{D^\bullet(A, B)} D_A^\uparrow(M^\bullet) \cong \text{Bar}^\bullet(A^{\text{opp}}, B^{\text{opp}}, M^\bullet) \otimes_A S_A$$

as a graded A^\sharp -module (with a forgotten differential). One can check directly that this isomorphism is an isomorphism of complexes. Finally we have

$$RHS = \text{Hom}_{A^\sharp}^\bullet(\text{Bar}^\bullet(A^\sharp, N^{\text{opp}}, M'^\bullet), \text{Bar}^\bullet(A^{\text{opp}}, B^{\text{opp}}, M^\bullet) \otimes_A S_A) = LHS.$$

Finally by Lemma 3.2.6

$$\mathrm{Ext}_A^{\frac{\infty}{2}+\bullet}(M'^{\bullet}, M^{\bullet}) = H^{\bullet}\left(\mathrm{Hom}_{D^{\bullet}(A,B)^{\mathrm{opp}}}(D^{\bullet}(A,B)^{\mathrm{opp}}, D^{\bullet}(N,k)^{\mathrm{opp}}, D_{A^{\sharp}}^{\downarrow}(M'^{\bullet})), D_A^{\uparrow}(M^{\bullet})\right). \quad \square$$

4. THE ALGEBRA A^{\sharp} FOR UNIVERSAL ENVELOPING ALGEBRAS

Consider a graded Lie algebra $\mathfrak{g} = \bigoplus_{n \in \mathbf{Z}} \mathfrak{g}_n$, $\dim \mathfrak{g}_n < \infty$. Suppose that \mathfrak{g} has a one dimensional central extension $\hat{\mathfrak{g}}$, i. e. $\hat{\mathfrak{g}} = \mathfrak{g} \oplus kC$ as a vector space and the element C of degree zero is central in $\hat{\mathfrak{g}}$.

The bracket in $\hat{\mathfrak{g}}$ is given by the formula

$$[g_1, g_2] = [g_1, g_2]_{\mathfrak{g}} + \omega(g_1, g_2)C, \quad [g, C] = 0, \quad g_1, g_2, g \in \mathfrak{g},$$

where ω is some 2-cocycle of the Lie algebra \mathfrak{g} .

4.1. Consider an associative algebra $A := U(\hat{\mathfrak{g}})/\{C - 1\}$. Here $U(\hat{\mathfrak{g}})$ denotes the universal enveloping algebra of the Lie algebra $\hat{\mathfrak{g}}$.

Then the grading on $\hat{\mathfrak{g}}$ induces gradings both on its universal enveloping algebra and on the algebra A as the relation $C = 1$ is homogeneous. Fix the natural triangular decomposition: $A = B \otimes N$ as a vector space, where

$$\mathfrak{b} := \bigoplus_{n \leq 0} \mathfrak{g}_n, \quad B := U(\mathfrak{b}), \quad \mathfrak{n} := \bigoplus_{n > 0} \mathfrak{g}_n, \quad N := U(\mathfrak{n}).$$

Note that the algebra A with this triangular decomposition satisfies the conditions 2.1. Our main task in this section is to describe the algebra A^{\sharp} for the algebra A explicitly.

4.2. First recall the construction of the critical cocycle of the Lie algebra \mathfrak{g} .

4.2.1. Let $V = \bigoplus_{n \in \mathbf{Z}} V_n$ be a graded vector space such that $\dim V_n < \infty$ for $n > 0$. Denote the space $\bigoplus_{n \leq 0} V_n$ (resp. the space $\bigoplus_{n > 0} V_n$) by V_- (resp. by V_+). Consider the Lie algebra $\mathfrak{gl}(V)$ of linear transformations of V satisfying the condition:

$$a \in \mathfrak{gl}(V), \quad a = (a_{ij}), \implies \exists m \in \mathbf{N} : a_{ij} = 0 \text{ for } |i - j| > m.$$

$\mathfrak{gl}(V)$ has a well known central extension given by the cocycle ω_0 :

$$\omega_0(a_1, a_2) = \mathrm{tr}(\pi \circ [a_1, a_2] - [\pi \circ a_1, \pi \circ a_2]).$$

Here π denotes the projection $V \longrightarrow V_+$ with the kernel V_- . Note that the trace is well defined on maps $V_+ \longrightarrow V_+$.

4.2.2. The adjoint representation of \mathfrak{g} provides the Lie algebra morphism $\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$. The inverse image of the 2-cocycle ω_0 is called the critical cocycle of the Lie algebra \mathfrak{g} (see e. g. [FFr]). We also denote it by ω_0 .

4.3. Now we introduce an analogue of the DG-algebra $D^{\bullet}(A, B)$. As before denote the left A -module $A \otimes_N \underline{k}$ by \underline{B} .

4.3.1. Consider a complex $D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b}) := \Lambda(\mathfrak{n}^*) \otimes \underline{B}$ with the differential defined as follows:

$$\begin{aligned} d : \Lambda^n(\mathfrak{n}^*) \otimes \underline{B} &\longrightarrow \Lambda^{n+1}(\mathfrak{n}^*) \otimes \underline{B} : a_1, \dots, a_{n+1} \in \mathfrak{n}, f : \Lambda^n(\mathfrak{n}) \longrightarrow \underline{B}, \\ df(a_1 \wedge \dots \wedge a_{n+1}) &:= \sum_{s=1}^{n+1} (-1)^s a_s \cdot f(a_1 \wedge \dots \wedge a_{s-1} \wedge a_{s+1} \wedge \dots \wedge a_{n+1}) \\ &+ \sum_{s < t} (-1)^{s+t} f([a_s, a_t] \wedge a_1 \wedge \dots \wedge a_{s-1} \wedge a_{s+1} \wedge \dots \wedge a_{t-1} \wedge a_{t+1} \wedge \dots \wedge a_{n+1}). \end{aligned}$$

We define a structure of the universal enveloping algebra of a graded Lie superalgebra on $D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b})$.

Denote by $\underline{\mathfrak{n}}$ the representation of the Lie algebra \mathfrak{b} in the vector space $\mathfrak{n} = \mathfrak{g}/\mathfrak{b}$. The dual representation is denoted by $\underline{\mathfrak{n}}^*$.

4.3.2. Consider a graded Lie superalgebra $\mathfrak{a} = \mathfrak{a}^0 \oplus \mathfrak{a}^1$, where $\mathfrak{a}^0 = \mathfrak{b}$, $\mathfrak{a}^1 = \underline{\mathfrak{n}}^*$, the bracket in \mathfrak{a} is defined as follows: $\mathfrak{a}^0 \wedge \mathfrak{a}^0 \longrightarrow \mathfrak{a}^0$ is just the bracket in \mathfrak{b} , $\mathfrak{a}^0 \otimes \mathfrak{a}^1 \longrightarrow \mathfrak{a}^1$ is given by the representation of \mathfrak{b} in $\underline{\mathfrak{n}}^*$.

Evidently $U(\mathfrak{a}) \cong D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b})$ as a graded vector space.

4.3.3. **Lemma:** The differential in $D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b})$ satisfies the Leibnitz rule. \square

4.3.4. Consider a morphism of DG-algebras $D^\bullet(A, B) \longrightarrow D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b})$ defined as follows. Consider first the canonical DG-algebra map $D^\bullet(U(\mathfrak{n}), k) \longrightarrow D^\bullet(\mathfrak{n}, 0)$ provided by the inclusion $\Lambda(\mathfrak{n}) \hookrightarrow T(\mathfrak{n})$:

$$n_1 \wedge \dots \wedge n_s \mapsto \sum_{\sigma} \text{sgn}(\sigma) n_{\sigma(n_1)} \otimes \dots \otimes n_{\sigma(n_s)},$$

where $\text{sgn}(\sigma)$ denotes the sign of the transposition σ . Note that the morphism commutes with differentials as it is induced by the morphism of standard resolutions — the bar resolution and the Lie one:

$$\begin{aligned} U(\mathfrak{n}) \otimes \Lambda(\mathfrak{n}) \otimes \underline{k} &\longrightarrow \text{Bar}^\bullet(U(\mathfrak{n}), k, \underline{k}), \\ \text{Hom}_{U(\mathfrak{n})}^\bullet(\text{Bar}^\bullet(U(\mathfrak{n}), k, \underline{k}), \underline{k}) &\longrightarrow \text{Hom}_{U(\mathfrak{n})}^\bullet(U(\mathfrak{n}) \otimes \Lambda(\mathfrak{n}) \otimes \underline{k}, \underline{k}). \end{aligned}$$

Now the induction functor provides the required morphism of DG-algebras

$$\begin{aligned} \varphi : D^\bullet(A, B) \cong \text{Hom}_A^\bullet(\text{Ind}_N^A \text{Bar}^\bullet(N, k, \underline{k}), \text{Ind}_N^A \underline{k}) \\ \longrightarrow \text{Hom}_A^\bullet(\text{Ind}_N^A (N \otimes \Lambda(\mathfrak{n}) \otimes \underline{k}), \text{Ind}_N^A \underline{k}) \cong D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b}). \end{aligned}$$

4.3.5. **Lemma:** φ is a quasiisomorphism of DG-algebras. \square

Remark: Both the construction of $D^\bullet(A, B)$ and the one of $D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b})$ are particular cases of the general quadratic-linear Koszul duality construction over a base ring due to L. Positselsky [P]. From that point of view the DG-algebra $D^\bullet(A, B)$ is the dual object for the algebra A equipped with a filtration induced by the filtration $k \subset N$ on N , and $D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b})$ is the dual object for the algebra A equipped with a filtration induced by the standard PBW filtration on $N = U(\mathfrak{n})$. The obvious morphism of filtered algebras leads to the morphism of the dual objects φ .

4.3.6. **Corollary:** (see 3.4.7) Let \underline{B} be the left $D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b})$ DG-module $\text{Ind}_{D^\bullet(\mathfrak{n}, 0)}^{D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b})} \underline{k}$. Then A coincides is isomorphic to the subalgebra in $\text{Ext}_{D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b})}^0(\underline{B}, \underline{B})$ generated by the images of B and N . \square

In particular the algebra A^\sharp can be recovered not only from $D^\bullet(A, B)^{\text{opp}}$ but also from $D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b})^{\text{opp}}$. Now we calculate the DG-algebra $D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b})^{\text{opp}}$.

4.4. Consider the antipode map in the Lie superalgebra \mathfrak{a} :

$$\alpha : U(\mathfrak{a}) \longrightarrow U(\mathfrak{a})^{\text{opp}}, \quad \alpha(x) = -x, \quad \alpha(y) = y \text{ for } x \in \mathfrak{a}^0, \quad y \in \mathfrak{a}^1.$$

Note that α need not necessarily commute with the differential in $U(\mathfrak{a})$. So to describe the algebra $D^\bullet(\hat{\mathfrak{g}}, \mathfrak{b})^{\text{opp}}$ it is sufficient to calculate relations between α and the differential.

Choose a homogeneous basis $\{x_m^{(i)}\}$ in \mathfrak{g} , here $x_m^{(i)} \in \mathfrak{g}_i$. In particular the set $\{x_m^{(j)}\}_{j \leq 0}$ forms a basis in \mathfrak{b} , the set $\{x_m^{(j)}\}_{j > 0}$ forms a basis in \mathfrak{n} . Let $\{x_m^{(j)*}\}_{j > 0}$ be the dual basis in $\mathfrak{n}^* = \sum_{j > 0} (\mathfrak{n}_j)^*$.

4.4.1. **Lemma:**

- (i) For $x \in \Lambda(\mathfrak{n}^*)$ the antipode map commutes with the differential;
- (ii) For $b \in \mathfrak{b}$ the antipode map satisfies the relation

$$\alpha \cdot d(b) = d \cdot \alpha(b) - \sum_{n \in \mathbf{Z}, i > 0} \omega_0(x_n^{(i)}, b) x_n^{(i)*}.$$

Proof. The first statement is proved by an obvious calculation. To prove the second one note that

$$\alpha \cdot d(b) = \sum_{m \in \mathbf{Z}, j > 0} \alpha(\pi_{\mathfrak{b}} \circ [x_m^{(j)}, b] \otimes x_m^{(j)*}) = d \cdot \alpha(b) + \sum_{m \in \mathbf{Z}, j > 0} (\pi_{\mathfrak{b}} \circ [x_m^{(j)}, b])(x_m^{(j)*}).$$

The second summand is a linear function on \mathfrak{n} . Let us calculate its value on a base vector $x_n^{(i)}$.

$$(\pi_{\mathfrak{b}} \circ [x_m^{(j)}, b]) x_m^{(j)*}(x_n^{(i)}) = x_m^{(j)*}([\pi_{\mathfrak{b}} \circ [x_m^{(j)}, b], x_n^{(i)}]) = \text{tr}(\pi_{\mathfrak{n}} \circ [x_n^{(i)}, \pi_{\mathfrak{b}} \circ [b, *]]).$$

The brackets here are the brackets in A , $\pi_{\mathfrak{n}}$ and $\pi_{\mathfrak{b}}$ denote the canonical projections onto \mathfrak{n} and \mathfrak{b} respectively. Thus the second summand is equal to

$$\sum_{n \in \mathbf{Z}, i > 0} \text{tr}(\pi_{\mathfrak{n}} \circ [x_n^{(i)}, \pi_{\mathfrak{b}} \circ [b, *]]) x_n^{(i)*} =: \sum_{n \in \mathbf{Z}, i > 0} \nu(x_n^{(i)}, b) x_n^{(i)*}.$$

On the other hand

$$\begin{aligned} \nu(x_n^{(i)}, b) &= \text{tr}(\pi_{\mathfrak{n}} \circ \text{ad}_{x_n^{(i)}}(\text{ad}_b - \pi_{\mathfrak{n}} \circ \text{ad}_b)) \\ &= -\text{tr}(\pi_{\mathfrak{n}} \circ ([\text{ad}_{x_n^{(i)}}, \text{ad}_b]) - [\pi_{\mathfrak{n}} \circ (\text{ad}_{x_n^{(i)}}), \pi_{\mathfrak{n}} \circ (\text{ad}_b)]) = -\omega_0(x_n^{(i)}, b). \quad \square \end{aligned}$$

4.4.2. **Corollary:** Let $\tilde{\mathfrak{g}}$ be the central extension of the Lie algebra \mathfrak{g} defined with the help of the cocycle $-\omega_0 - \omega$. Then the algebra A^\sharp is isomorphic to $U(\tilde{\mathfrak{g}})/\{C - 1\}$. \square

4.4.3. **Remark:** The proof of the fact that the algebra $\text{End}_{U(\mathfrak{g})}(S_{U(\mathfrak{g})})$ differs from $U(\mathfrak{g})$ by a Lie algebra 2-cocycle is contained essentially in [V]. The method used there is different from ours and is probably easier. Yet our proof follows from the general picture that explains the very appearance of the cocycle. In particular, when changed a little, our proof still holds in the case of affine quantum groups.

4.5. **Critical cocycle in the case of affine Lie algebras.** To know the bimodule structure on S_A one has to describe not only the cohomology class of the critical cocycle but the critical cocycle itself. In this section we present a direct calculation for the critical cocycle ω_0 in the case of affine Lie algebras. It turns out that the cocycle depends on the type of the triangular decomposition of the affine Lie algebra.

4.5.1. From now on we suppose that our base field is \mathbf{C} . Fix a Cartan matrix $(a_{ij})_{i,j=1}^r$ of the finite type and consider the corresponding semisimple Lie algebra $\bar{\mathfrak{g}}$. Recall that the affine Lie algebra \mathfrak{g} for the semisimple Lie algebra $\bar{\mathfrak{g}}$ is defined as a central extension of a loop algebra $\mathcal{L}\bar{\mathfrak{g}} := \bar{\mathfrak{g}} \otimes \mathbf{C}[t, t^{-1}]$. Namely, $\mathfrak{g} := \mathcal{L}\bar{\mathfrak{g}} \oplus \mathbf{C}K$, and the bracket in \mathfrak{g} is defined as follows:

$$[g_1 \otimes t^n, g_2 \otimes t^m] = [g_1, g_2] \otimes t^{n+m} + \delta_{n+m,0} n B(g_1, g_2) K,$$

where $g_1, g_2 \in \bar{\mathfrak{g}}$, $n, m \in \mathbf{Z}$, and $B(\cdot, \cdot)$ denotes the Killing form of $\bar{\mathfrak{g}}$.

Denote by \bar{R}, \bar{R}^+ and $\bar{\Sigma} = \{\alpha_1, \dots, \alpha_r\}$ the root system of $\bar{\mathfrak{g}}$, the set of positive roots and the set of simple roots respectively. Let X be the corresponding weight lattice, $(\cdot | \cdot)$ denotes the canonical symmetric bilinear form on X .

Thus $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^+ \oplus \mathfrak{h} \oplus \bar{\mathfrak{g}}^-$, where \mathfrak{h} denotes the Cartan subalgebra in $\bar{\mathfrak{g}}$, and

$$\bar{\mathfrak{g}}^+ = \bigoplus_{\alpha \in \bar{R}^+} \bar{\mathfrak{g}}_\alpha, \quad \bar{\mathfrak{g}}^- = \bigoplus_{\alpha \in \bar{R}^-} \bar{\mathfrak{g}}_\alpha.$$

Consider the Chevalley generators of the Lie algebra $\bar{\mathfrak{g}}$: $e_i \in \bar{\mathfrak{g}}_{\alpha_i}, f_i \in \bar{\mathfrak{g}}_{-\alpha_i}$, $i = 1, \dots, r$.

4.5.2. It is well known that the affine Lie algebra \mathfrak{g} is a Kac-Moody Lie algebra. Denote by R, R^+ and Σ the root system of $\bar{\mathfrak{g}}$, the set of positive roots and the set of simple roots respectively. Then $\Sigma = \bar{\Sigma} \cup \{\alpha_0\}$. Thus $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathbf{C}K \oplus \mathfrak{n}^-$, and

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha = \bar{\mathfrak{g}}^+ \oplus \bar{\mathfrak{g}} \otimes t\mathbf{C}[t], \quad \mathfrak{n}^- = \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha = \bar{\mathfrak{g}}^- \oplus \bar{\mathfrak{g}} \otimes t^{-1}\mathbf{C}[t^{-1}].$$

Recall the definition of the Chevalley generators for \mathfrak{g} . Set $e_{\alpha_i} := e_i$, $e_{-\alpha_i} := f_i$, $i = 1, \dots, r$, where e_i, f_i are the Chevalley generators for $\bar{\mathfrak{g}}$, $[e_i, f_i] = h_i$. Let α_{top} be the highest root of the root system \bar{R} . Choose a vector h_{top} in $[\bar{\mathfrak{g}}_{\alpha_{\text{top}}}, \bar{\mathfrak{g}}_{-\alpha_{\text{top}}}] \subset \mathfrak{h}$ such that $\alpha_{\text{top}}(h_{\text{top}}) = 2$. Fix $e_{\text{top}} \in \bar{\mathfrak{g}}_{\alpha_{\text{top}}}$ and $f_{\text{top}} \in \bar{\mathfrak{g}}_{-\alpha_{\text{top}}}$ such that $[e_{\text{top}}, f_{\text{top}}] = h_{\text{top}}$. Then set

$$e_{\alpha_0} := f_{\text{top}} \otimes t, \quad e_{-\alpha_0} := e_{\text{top}} \otimes t^{-1}.$$

4.5.3. **Lemma:** (see [K] (7.4.1))

$$[e_{\text{top}}, f_{\text{top}}] = \sum_{i=1}^r a_i^\vee h_i,$$

where $a_1^\vee, \dots, a_r^\vee$ are the marks for the dual Dynkin diagram. □

Consider the following two natural gradings on \mathfrak{g} and the corresponding triangular decompositions of \mathfrak{g} .

4.5.4. Let $\deg_{(1)}(g \otimes t^n) = n$. The corresponding triangular decomposition looks as follows:

$$\mathfrak{g} = \mathfrak{g}_{>0} \oplus \mathfrak{g}_{\leq 0}, \quad \mathfrak{g}_{>0} := \bar{\mathfrak{g}} \otimes t\mathbf{C}[t], \quad \mathfrak{g}_{\leq 0} := \bar{\mathfrak{g}} \otimes \mathbf{C}[t^{-1}].$$

4.5.5. Let $\deg_{(2)}$ be the grading on the Lie algebra \mathfrak{g} obtained from the root decomposition of \mathfrak{g} by putting

$$\deg_{(2)}(g_\alpha) = 1, \quad \text{where } g_\alpha \in \mathfrak{g}_\alpha, \text{ for any } \alpha \in \Sigma.$$

Then the corresponding triangular decomposition is as follows:

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{b}^-, \quad \mathfrak{b}^- := \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathbf{C}K.$$

4.5.6. **Remark:** More generally every parabolic subalgebra \mathfrak{p} in $\bar{\mathfrak{g}}$ with nilpotent radical $\mathfrak{u} \subset \bar{\mathfrak{g}}^+$ provides a triangular decomposition of \mathfrak{g} such that the positive Lie subalgebra is equal to $\mathfrak{u} \oplus \bar{\mathfrak{g}} \otimes t\mathbf{C}[t]$. Thus here we consider the cases of \mathfrak{p} equal to $\bar{\mathfrak{g}}$ and to the Borel subalgebra in $\bar{\mathfrak{g}}$.

Denote the critical cocycle that corresponds to the first (resp. the second) triangular decomposition by $\omega_0^{(1)}$ (resp. by $\omega_0^{(2)}$).

4.5.7. **Lemma:** Let ω be a 2-cocycle of the Lie algebra \mathfrak{g} that preserves the root grading. Then ω is completely defined by the set of its values $\{\omega(e_\alpha, e_{-\alpha}) | \alpha \in \Sigma\}$.

Proof. By definition of a 2-cocycle for any $g_1, g_2, g_3 \in \mathfrak{g}$

$$\omega([g_1, g_2], g_3) = \omega(g_1, [g_2, g_3]).$$

Moreover the cocycle ω is nontrivial on a pair of homogeneous elements $g_1, g_2 \in \mathfrak{g}$ only if $g_1 \in \mathfrak{g}_\alpha, g_2 \in \mathfrak{g}_{-\alpha}, \alpha \in R$ or both g_1 and g_2 belong to $\mathfrak{h} \oplus \mathbf{CK}$. By definition of the affine algebra for any $\alpha \in R$ $\dim \mathfrak{g}_\alpha = 1$, and for any $\alpha \in R^+$ there exist $\beta_1, \dots, \beta_s \in \Sigma$ such that

$$e_\alpha := [e_{\beta_1}, [e_{\beta_2}, \dots, [e_{\beta_{s-1}}, e_{\beta_s}] \dots]] \neq 0 \in \mathfrak{g}_\alpha$$

(resp. for any $\alpha \in R^-$ there exist $\beta_1, \dots, \beta_s \in \Sigma$ such that

$$e_{-\alpha} := [e_{-\beta_1}, [e_{-\beta_2}, \dots, [e_{-\beta_{s-1}}, e_{-\beta_s}] \dots]] \neq 0 \in \mathfrak{g}_\alpha).$$

The vector space \mathfrak{h} has a base consisting of $h_i := [e_{\alpha_i}, e_{-\alpha_i}], \alpha_i \in \bar{\Sigma}$. The space \mathbf{CK} is generated by the vector $[h_1 \otimes t, h_1 \otimes t^{-1}]$. Note that $h_1 \otimes t \in \mathfrak{n}^+, h_1 \otimes t^{-1} \in \mathfrak{n}^-$. Now the statement of the Lemma is proved by induction by the length of the expression of the homogeneous element g_1 via the generators of \mathfrak{g} . \square

Consider the 1-cochain ρ of \mathfrak{g} defined as follows:

$$\rho(g_\alpha) = 0, g_\alpha \in \mathfrak{g}_\alpha, \rho(K) = 0, \rho(h_i) = 1, i = 1, \dots, r.$$

Then for every $\alpha_i \in \bar{\Sigma}$ $d\rho(e_{\alpha_i}, e_{-\alpha_i}) = 1$.

Recall that the dual Coxeter number h^\vee of the root system \bar{R} is equal to $a_1^\vee + \dots + a_r^\vee + 1$.

4.5.8. **Lemma:** $d\rho(e_{\alpha_0}, e_{-\alpha_0}) = h^\vee - 1$.

Proof. Follows immediately from 4.5.3. \square

Denote by ν the canonical 2-cocycle of \mathfrak{g}

$$(g_1 \otimes t^k, g_2 \otimes t^m) \mapsto n\delta_{k+m,0}(g_1, g_2), g_1, g_2 \in \bar{\mathfrak{g}}, k, m \in \mathbf{Z},$$

where (\cdot, \cdot) denotes the normalized invariant bilinear form on $\bar{\mathfrak{g}}$:

$$(\cdot, \cdot) := \frac{1}{2h^\vee} B(\cdot, \cdot).$$

4.5.9. **Proposition:**

$$(i) \quad \omega_0^{(1)} = 2h^\vee \nu$$

$$(ii) \quad \omega_0^{(2)} = 2h^\vee \nu + 2d\rho.$$

Proof. (i) Since the cocycle $\omega_0^{(1)}$ preserves the first grading, for every $g_1, g_2 \in \bar{\mathfrak{g}}$

$$\omega_0^{(1)}(g_1 \otimes t, g_2 \otimes t^{-1}) = \text{tr}_{\bar{\mathfrak{g}} \otimes t}(\text{ad}(g_1 \otimes t) \text{ad}(g_2 \otimes t^{-1})) = B(g_1, g_2).$$

(ii) To calculate $\omega_0^{(2)}(e_{\alpha_i}, e_{-\alpha_i})$, $i = 1, \dots, r$ note that as the cocycle preserves the root grading, the only nontrivial summand in the root sum will be

$$\langle e_{\alpha_i}^*, [e_{\alpha_i}, [e_{-\alpha_i}, e_{\alpha_i}]] \rangle.$$

To calculate $\omega_0^{(2)}(e_{\alpha_0}, e_{-\alpha_0})$ note that the only root vector e in $\bar{\mathfrak{g}} \otimes t$ such that $[e_{-\alpha_0}, e] \in \mathfrak{b}^-$ is e_{α_0} . \square

4.5.10. Fix a level k and a character λ of \mathfrak{h} (resp a finite dimensional representation of $\bar{\mathfrak{g}} L(\mu)$). Recall that the module $M(\lambda) := \text{Ind}_{U(\mathfrak{n}^+ \oplus \mathfrak{h})}^{U_k(\mathfrak{g})} \mathbf{C}(\lambda)$ is called the Verma module, and the module $W(\mu) := \text{Ind}_{U(\mathfrak{g}_{\geq 0})}^{U_k(\mathfrak{g})} L(\mu)$ is called the Weyl module over \mathfrak{g} on the level k .

Remark: Comparing the previous calculation with 3.5.3 we obtain in particular the following statement:

- (i) The additive categories generated by Verma modules over \mathfrak{g} on levels k and $-2h^\vee - k$ are antiequivalent;
- (ii) The additive categories generated by Weyl modules over \mathfrak{g} on levels k and $-2h^\vee - k$ are antiequivalent.

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